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Analogues of Discrete Torsion for the M-Theory Three-Form

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In this article we shall outline a derivation of the analogue of discrete torsion for the M-theory three-form potential. We find that some of the differences between orbifold group actions on the C field are classified by $H^3(\Gamma, U(1))$. We also compute the phases that the low-energy effective action of a membrane on T^3 would see in the analogue of a twisted sector, and note that they are invariant under the obvious $SL(3, \mathbf{Z})$ action.

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1 Introduction

Discrete torsion is an obscure-looking degree of freedom associated with orbifolds. It was originally discovered as a set of complex phases that could be added to twisted-sector contributions to one-loop partition functions [1], and via the constraint of modular invariance, was found to be classified by $H^2(\Gamma, U(1))$. Although a number of papers have been written on the subject (see for example [2, 3, 4]), it was not until quite recently in [5, 6, 7] that a basic geometric understanding of discrete torsion was worked out.

In a nutshell,

Discrete torsion is the choice of orbifold group action on the B field.

More generally, in any theory containing fields with gauge invariances, to define an orbifold it does not suffice to only define the orbifold action on the base space. One must also specify the orbifold action on the fields. The choice of orbifold action on the fields is not unique, precisely because of the possibility of combining the orbifold group action with a gauge transformation.

For vector fields, these degrees of freedom lead to orbifold Wilson lines. For B fields, these degrees of freedom lead to discrete torsion.

Put another way, discrete torsion can be understood in purely mathematical terms, without any reference to string theory – it has nothing to do with string theory *per se*.

One can now ask, what are the corresponding degrees of freedom for the other tensor field potentials appearing in string theory? For example, consider the three-form potential C in M theory. What degrees of freedom does one have when defining a quotient?

In this paper we describe a preliminary attack on this question. More precisely, we work out the degrees of freedom describing possible orbifold group actions on a C field whose curvature is the image of an element of $H^4(\mathbf{Z})$. Although this sounds just right to describe the M-theory three-form, we must caution the reader that we are almost certainly overlooking important physical subtleties. For example, we will only consider the C field in isolation, whereas it is known [8] that there are interaction terms which shift the quantization of the curvature of the C field. To get a physically correct answer, we would probably need to take into account that interaction, which we shall not do. Also, it has recently been pointed out [9, 10] that, in addition to the gravitational correction just discussed, the M-theory C form potential should also be understood as the Chern-Simons 3-form on an E_8 gauge bundle with connection. In order to properly study orbifold group actions on the M-theory three-form C, one would need to study orbifold group actions on Chern-Simons forms induced by orbifold group actions on E_8 bundles with connection, which we shall not do.

Despite the fact that the results in this paper may well not be immediately applicable to the M-theory three-form, we think it is extremely important to make the point that there exists an analogue of discrete torsion for the M-theory three-form and the other tensor field potentials of string theory, and that techniques exist that, in principle, allow one to calculate these degrees of freedom.

In section 2, we study orbifold group actions on a C field. We find that the group $H^3(\Gamma, U(1))$ arises in describing the differences between some of the orbifold group actions, just as the group $H^2(\Gamma, U(1))$ arose in [5, 6, 7] in describing the difference between some of the orbifold group actions on the B field. We also find additional possible orbifold group actions, beyond those described by $H^3(\Gamma, U(1))$.

In section 3 we work out the analogue of twisted sector phases for membranes. Specifically, just as the presence of the term $\int B$ in the string sigma model leads to the twisted sector phases that originally characterized discrete torsion for B fields [5, 7], since membrane low-energy effective actions have a term $\int C$ one expects to have twisted sector phases associated with membranes. We compute these phases for membranes on T^3 (described as an open box on the covering space, with sides identified by the orbifold group action), and also check that these phases are invariant under the natural $SL(3, \mathbf{Z})$ action on T^3, the analogue of the modular invariance constraint on the phases of B field discrete torsion. (We do not wish to imply that modular invariance is necessarily a sensible notion for low-energy effective actions on membrane worldvolumes, but it is certainly amusing to check that the phases we derive from more fundamental considerations also happen to be $SL(3, \mathbf{Z})$-invariant.)

Our methods for understanding analogues of discrete torsion for the M-theory three-form potential C are purely mathematical. How would such degrees of freedom show up physically? If we could compactify M-theory on an additional S^1, for example, how would these degrees of freedom be observed in string perturbation theory? If they could be observed at all in string perturbation theory, one would surely need to be doing string perturbation theory in a framework in which Ramond-Ramond fields could be easily analyzed, as for example [11]. However, it is not at all clear that such degrees of freedom can be observed in string perturbation theory. For example, although $H^3(\Gamma, U(1))$ enters naturally into "membrane twisted sectors," as we shall see later, a string worldsheet is one dimension too small to naturally couple to these degrees of freedom.

In principle one could also calculate analogues of discrete torsion for the type II Ramond-Ramond tensor field potentials directly. As these fields are presently believed to be described in terms of K-theory, one would need a Cheeger-Simons-type description of K-theory. Unfortunately, no such description has yet been published, so we shall not attack the problem of such discrete torsion analogues here.

The methods in this paper are a direct outgrowth of methods used in [7], and we shall assume the reader possesses a working knowledge of that reference.

2 Orbifold group actions on 2-gerbes

In this section we shall classify orbifold group actions on three-form potentials, as described in local coordinate patches. Although we shall often speak of such potentials as "connections on 2-gerbes," in fact we shall not deal at all with 2-gerbes, but shall merely work in a local coordinate description.

As described several times earlier, to describe the action of an orbifold group on a theory containing fields with gauge symmetries, it is not sufficient merely to describe the action of the orbifold group on the base space. In general, one can combine the action of the orbifold group with gauge transformations, and so one has to also be careful to specify the action of the orbifold group on any fields with gauge symmetries.

2.1 The set of orbifold group actions

Let $\{U_\alpha\}$ be a "good invariant" cover, as discussed in [5, 7]. Then, a three-form field potential is described as a collection of three-forms C^α, one for each open set U_α, related by gauge transformation on overlaps. More precisely,

$$\begin{aligned} C^\alpha \,-\, C^\beta &= dB^{\alpha\beta} \\ B^{\alpha\beta} \,+\, B^{\beta\gamma} \,+\, B^{\gamma\alpha} &= dA^{\alpha\beta\gamma} \\ A^{\beta\gamma\delta} \,-\, A^{\alpha\gamma\delta} \,+\, A^{\alpha\beta\delta} \,-\, A^{\alpha\beta\gamma} &= d\log h_{\alpha\beta\gamma\delta} \\ \delta h_{\alpha\beta\gamma\delta} &= 1 \end{aligned}$$

(It should be mentioned that the expressions above do not introduce any new structure not already present in physics. Readers might object that they are only acquainted with a C field in physics, and do not recall seeing an associated B or A as above; however, such readers should be reminded that the C fields on distinct open patches are related by *some* gauge transformation B – we are merely making that patch overlap information explicit, rather than leaving it implicit as is usually done.) We have implicitly assumed, in writing the above, that the exterior derivative of the curvature of C vanishes – that there are no magnetic sources present – and that the curvature of C is the image of an element of $H^4(\mathbf{Z})$, i.e., that the curvature has integral periods.

We shall begin by studying the behavior of the $h_{\alpha\beta\gamma\delta}$ under pullbacks. Let $g \in \Gamma$, then $g^* h_{\alpha\beta\gamma\delta}$ and $h_{\alpha\beta\gamma\delta}$ should differ by only a coboundary:

$$g^* h_{\alpha\beta\gamma\delta} \;=\; \left(h_{\alpha\beta\gamma\delta}\right) \left(\nu^g_{\beta\gamma\delta}\right) \left(\nu^g_{\alpha\gamma\delta}\right)^{-1} \left(\nu^g_{\alpha\beta\delta}\right) \left(\nu^g_{\alpha\beta\gamma}\right)^{-1} \tag{1}$$

If this were not true, one could not even begin to make sense of an orbifold action, as the C field would not be even remotely symmetric under the orbifold group action. Given this

assumption, we shall now follow a self-consistent bootstrap in the spirit of [7] to work out the orbifold group action on the other data defining the C field.

Now, compare the pullback of $h_{\alpha\beta\gamma\delta}$ by the product $g_1 g_2$ ($g_1, g_2 \in \Gamma$) and by each group element separately:

$$\begin{aligned}(g_1 g_2)^* h_{\alpha\beta\gamma\delta} &= (h_{\alpha\beta\gamma\delta}) \left(\nu^{g_1 g_2}_{\beta\gamma\delta}\right) \left(\nu^{g_1 g_2}_{\alpha\gamma\delta}\right)^{-1} \cdots \\ g_2^* g_1^* h_{\alpha\beta\gamma\delta} &= (h_{\alpha\beta\gamma\delta}) \left(\nu^{g_2}_{\beta\gamma\delta}\, g_2^* \nu^{g_1}_{\beta\gamma\delta}\right) \left(\nu^{g_2}_{\alpha\gamma\delta}\, g_2^* \nu^{g_1}_{\alpha\gamma\delta}\right)^{-1} \cdots\end{aligned}$$

from which we derive[1] the condition that $\nu^{g_1 g_2}$ and $\nu^{g_2} g_2^* \nu^{g_1}$ must agree, up to a coboundary:

$$\nu^{g_1 g_2}_{\alpha\beta\gamma} = \left(\nu^{g_2}_{\alpha\beta\gamma}\right) \left(g_2^* \nu^{g_1}_{\alpha\beta\gamma}\right) \lambda^{g_1,g_2}_{\alpha\beta}\, \lambda^{g_1,g_2}_{\beta\gamma}\, \lambda^{g_1,g_2}_{\gamma\alpha} \tag{2}$$

Next, we need to examine the coboundaries $\lambda^{g_1,g_2}_{\alpha\beta}$ more carefully. There are two distinct ways to relate $\nu^{g_1 g_2 g_3}$ and $\nu^{g_3} g_3^* (\nu^{g_2} g_2^* \nu^{g_1})$, and they must agree, which constrains the λ^{g_1,g_2}. Specifically,

$$\begin{aligned}\nu^{g_1 g_2 g_3}_{\alpha\beta\gamma} &= \left(\nu^{g_3}_{\alpha\beta\gamma}\right) \left(g_3^* \nu^{g_1 g_2}_{\alpha\beta\gamma}\right) \lambda^{g_1 g_2, g_3}_{\alpha\beta}\, \lambda^{g_1 g_2, g_3}_{\beta\gamma}\, \lambda^{g_1 g_2, g_3}_{\gamma\alpha} \\ &= \left(\nu^{g_3}_{\alpha\beta\gamma}\right) g_3^* \left(\nu^{g_2}_{\alpha\beta\gamma}\, g_2^* \nu^{g_1}_{\alpha\beta\gamma}\, \lambda^{g_1,g_2}_{\alpha\beta}\, \lambda^{g_1,g_2}_{\beta\gamma}\, \lambda^{g_1,g_2}_{\gamma\alpha}\right) \lambda^{g_1 g_2, g_3}_{\alpha\beta}\, \lambda^{g_1 g_2, g_3}_{\beta\gamma}\, \lambda^{g_1 g_2, g_3}_{\gamma\alpha} \\ \text{also} &= \left(\nu^{g_2 g_3}_{\alpha\beta\gamma}\right) \left((g_2 g_3)^* \nu^{g_1}_{\alpha\beta\gamma}\right) \lambda^{g_1, g_2 g_3}_{\alpha\beta}\, \lambda^{g_1, g_2 g_3}_{\beta\gamma}\, \lambda^{g_1, g_2 g_3}_{\gamma\alpha} \\ &= \left(\nu^{g_3}_{\alpha\beta\gamma}\, g_3^* \nu^{g_2}_{\alpha\beta\gamma}\, \lambda^{g_2,g_3}_{\alpha\beta}\, \lambda^{g_2,g_3}_{\beta\gamma}\, \lambda^{g_2,g_3}_{\gamma\alpha}\right) \left((g_2 g_3)^* \nu^{g_1}_{\alpha\beta\gamma}\right) \lambda^{g_1, g_2 g_3}_{\alpha\beta}\, \lambda^{g_1, g_2 g_3}_{\beta\gamma}\, \lambda^{g_1, g_2 g_3}_{\gamma\alpha}\end{aligned}$$

In order for the equations above to be consistent, we demand that $\lambda^{g_1 g_2, g_3} g_3^* \lambda^{g_1,g_2}$ and $\lambda^{g_1, g_2 g_3} \lambda^{g_2,g_3}$ agree up to a coboundary:

$$\left(\lambda^{g_1 g_2, g_3}_{\alpha\beta}\right) \left(g_3^* \lambda^{g_1,g_2}_{\alpha\beta}\right) = \left(\lambda^{g_1, g_2 g_3}_{\alpha\beta}\right) \left(\lambda^{g_2,g_3}_{\alpha\beta}\right) \left(\gamma^{g_1,g_2,g_3}_{\alpha}\right) \left(\gamma^{g_1,g_2,g_3}_{\beta}\right)^{-1} \tag{3}$$

Finally, from consistency of the relation (3) in studying products of four group elements, we can derive a constraint on the coboundaries γ^{g_1,g_2,g_3}. Specifically, by solving equation (3) for $(g_3 g_4)^* \lambda^{g_1,g_2}$ in two distinct ways, one can derive

$$\left(\gamma^{g_1,g_2,g_3 g_4}_{\alpha}\right) \left(\gamma^{g_1 g_2, g_3, g_4}_{\alpha}\right) = \left(\gamma^{g_1, g_2 g_3, g_4}_{\alpha}\right) \left(\gamma^{g_2,g_3,g_4}_{\alpha}\right) \left(g_4^* \gamma^{g_1,g_2,g_3}_{\alpha}\right) \tag{4}$$

So far we have derived the form of a lift of Γ to merely the Čech cocycles $h_{\alpha\beta\gamma\delta}$; we still need to describe how the orbifold group acts on the local three-forms, two-forms, and one-forms defining the connection on the 2-gerbe.

[1] The attentive reader should, correctly, object that the equation (2) is slightly stronger than implied by self-consistency of $(g_1 g_2)^* h_{\alpha\beta\gamma\delta}$ – that only tells us that the Čech coboundary of $\nu^{g_1 g_2}$ must equal the Čech coboundary of $\nu^{g_2} g_2^* \nu^{g_1}$. As in [7] we are using self-consistency to generate constraints of the form of equation (2) on Čech cochains, not coboundaries thereof. If the reader prefers, we are using self-consistency to generate an ansatz.

We can write

$$
\begin{aligned}
g^* C^\alpha &= C^\alpha + C(g)^\alpha \\
g^* B^{\alpha\beta} &= B^{\alpha\beta} + B(g)^{\alpha\beta} \\
g^* A^{\alpha\beta\gamma} &= A^{\alpha\beta\gamma} + A(g)^{\alpha\beta\gamma}
\end{aligned}
$$

for some local three-forms $C(g)^\alpha$, two-forms $B(g)^{\alpha\beta}$, and one-forms $A(g)^{\alpha\beta\gamma}$. Using self-consistency, we shall derive more meaningful expressions for $C(g)^\alpha$, $B(g)^{\alpha\beta}$, and $A(g)^{\alpha\beta\gamma}$.

Expand both sides of the expression

$$g^* \left(C^\alpha - C^\beta \right) = g^* dB^{\alpha\beta}$$

to show that

$$C(g)^\alpha - C(g)^\beta = dB(g)^{\alpha\beta\gamma} \tag{5}$$

Similarly, use

$$g^* \left(B^{\alpha\beta} + B^{\beta\gamma} + B^{\gamma\alpha} \right) = dA^{\alpha\beta\gamma}$$

to show

$$B(g)^{\alpha\beta} + B(g)^{\beta\gamma} + B(g)^{\gamma\alpha} = dA(g)^{\alpha\beta\gamma} \tag{6}$$

and use

$$g^* \left(\delta A^{\alpha\beta\gamma} \right) = g^* d \log h_{\alpha\beta\gamma\delta}$$

to show

$$A(g)^{\alpha\beta\gamma} = d \log \nu^g_{\alpha\beta\gamma} + \Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(1)}(g)^{\beta\gamma} + \Lambda^{(1)}(g)^{\gamma\alpha} \tag{7}$$

for some local one-forms $\Lambda^{(1)}(g)^{\alpha\beta}$. Plugging this expression back into equation (6), we find

$$B(g)^{\alpha\beta} = d\Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(2)}(g)^\alpha - \Lambda^{(2)}(g)^\beta \tag{8}$$

for some local two-forms $\Lambda^{(2)}(g)^\alpha$. Plugging this back into equation (5), we find

$$C(g)^\alpha = d\Lambda^{(2)}(g)^\alpha \tag{9}$$

To summarize progress so far, we have found

$$
\begin{aligned}
g^* C^\alpha &= C^\alpha + d\Lambda^{(2)}(g)^\alpha \\
g^* B^{\alpha\beta} &= B^{\alpha\beta} + d\Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(2)}(g)^\alpha - \Lambda^{(2)}(g)^\beta \\
g^* A^{\alpha\beta\gamma} &= A^{\alpha\beta\gamma} + d \log \nu^g_{\alpha\beta\gamma} + \Lambda^{(1)}(g)^{\alpha\beta} + \Lambda^{(1)}(g)^{\beta\gamma} + \Lambda^{(1)}(g)^{\gamma\alpha}
\end{aligned}
$$

for some forms $\Lambda^{(1)}(g)^{\alpha\beta}$ and $\Lambda^{(2)}(g)^\alpha$.

Next, we need to determine how $\Lambda^{(1)}(g_1 g_2)^{\alpha\beta}$ and $\Lambda^{(2)}(g_1 g_2)^\alpha$ are related to $\Lambda^{(1)}(g_1)^{\alpha\beta}$, $\Lambda^{(1)}(g_2)^{\alpha\beta}$, $\Lambda^{(2)}(g_1)^\alpha$, and $\Lambda^{(2)}(g_2)^\alpha$.

By evaluating $(g_1 g_2)^* C^\alpha$ in two different ways, we find

$$\Lambda^{(2)}(g_1 g_2)^\alpha \;=\; \Lambda^{(2)}(g_2)^\alpha \;+\; g_2^* \Lambda^{(2)}(g_1)^\alpha \;+\; d\Lambda^{(3)}(g_1, g_2)^\alpha \tag{10}$$

for some local one-forms $\Lambda^{(3)}(g_1, g_2)^\alpha$.

By evaluating $(g_1 g_2)^* A^{\alpha\beta\gamma}$ in two different ways, we find

$$\delta \left(d \log \lambda_{\alpha\beta}^{g_1, g_2} \;+\; \Lambda^{(1)}(g_1 g_2)^{\alpha\beta} \right) \;=\; \delta \left(\Lambda^{(1)}(g_2)^{\alpha\beta} \;+\; g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} \right)$$

By evaluating $(g_1 g_2)^* B^{\alpha\beta\gamma}$ in two different ways, we find

$$d \left(\Lambda^{(1)}(g_1 g_2)^{\alpha\beta} \;+\; \Lambda^{(3)}(g_1, g_2)^\alpha \;-\; \Lambda^{(3)}(g_1, g_2)^\beta \right) \;=\; d \left(\Lambda^{(1)}(g_2)^{\alpha\beta} \;+\; g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} \right)$$

After combining these two equations, we find

$$\Lambda^{(1)}(g_1 g_2)^{\alpha\beta} \;+\; \Lambda^{(3)}(g_1, g_2)^\alpha \;-\; \Lambda^{(3)}(g_1, g_2)^\beta \;+\; d \log \lambda_{\alpha\beta}^{g_1, g_2} \;=\; \Lambda^{(1)}(g_2)^{\alpha\beta} \;+\; g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} \tag{11}$$

By using equation (10) to evaluate $\Lambda^{(2)}(g_1 g_2 g_3)^\alpha$ in two different ways, we find

$$d \left(\Lambda^{(3)}(g_2, g_3)^\alpha \;+\; \Lambda^{(3)}(g_1, g_2 g_3)^\alpha \right) \;=\; d \left(g_3^* \Lambda^{(3)}(g_1, g_2)^\alpha \;+\; \Lambda^{(3)}(g_1 g_2, g_3)^\alpha \right)$$

By using equation (11) to evaluate $\Lambda^{(1)}(g_1 g_2 g_3)^{\alpha\beta}$ in two different ways, we find

$$\delta \left(\Lambda^{(3)}(g_2, g_3)^\alpha \;+\; \Lambda^{(3)}(g_1, g_2 g_3)^\alpha \right) \;=\; \delta \left(g_3^* \Lambda^{(3)}(g_1, g_2)^\alpha \;+\; \Lambda^{(3)}(g_1 g_2, g_3)^\alpha \;+\; d \log \gamma_\alpha^{g_1, g_2, g_3} \right)$$

Combining these two equations, we find

$$\Lambda^{(3)}(g_2, g_3)^\alpha \;+\; \Lambda^{(3)}(g_1, g_2 g_3)^\alpha \;=\; g_3^* \Lambda^{(3)}(g_1, g_2)^\alpha \;+\; \Lambda^{(3)}(g_1 g_2, g_3)^\alpha \;+\; d \log \gamma_\alpha^{g_1, g_2, g_3} \tag{12}$$

(In principle, we could add a term of the form

$$d\Lambda^{(4)}(g_1, g_2, g_3)^\alpha \;-\; d\Lambda^{(4)}(g_1, g_2, g_3)^\beta$$

to one side of the expression above, i.e., something annihilated by both d and δ; however, the only quantities that we shall use are $d\Lambda^{(3)}$ and $\delta\Lambda^{(3)}$, so we shall ignore such additional possible terms.)

The calculation above completes our derivation of how orbifold groups act on 2-gerbes with connection, at the level of Čech cocycles. We have included a table below to summarize the results:

$$
\begin{aligned}
g^* C^{\alpha} &= C^{\alpha} \,+\, d\Lambda^{(2)}(g)^{\alpha} \\
g^* B^{\alpha\beta} &= B^{\alpha\beta} \,+\, d\Lambda^{(1)}(g)^{\alpha\beta} \,+\, \Lambda^{(2)}(g)^{\alpha} \,-\, \Lambda^{(2)}(g)^{\beta} \\
g^* A^{\alpha\beta\gamma} &= A^{\alpha\beta\gamma} \,+\, d\log \nu^g_{\alpha\beta\gamma} \,+\, \Lambda^{(1)}(g)^{\alpha\beta} \,+\, \Lambda^{(1)}(g)^{\beta\gamma} \,+\, \Lambda^{(1)}(g)^{\gamma\alpha} \\
g^* h_{\alpha\beta\gamma\delta} &= \left(h_{\alpha\beta\gamma\delta} \right) \left(\nu^g_{\beta\gamma\delta} \right) \left(\nu^g_{\alpha\gamma\delta} \right)^{-1} \left(\nu^g_{\alpha\beta\delta} \right) \left(\nu^g_{\alpha\beta\gamma} \right)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\Lambda^{(2)}(g_1 g_2)^{\alpha} &= \Lambda^{(2)}(g_2)^{\alpha} \,+\, g_2^* \Lambda^{(2)}(g_1)^{\alpha} \,+\, d\Lambda^{(3)}(g_1, g_2)^{\alpha} \\
\Lambda^{(1)}(g_1 g_2)^{\alpha\beta} &= \Lambda^{(1)}(g_2)^{\alpha\beta} \,+\, g_2^* \Lambda^{(1)}(g_1)^{\alpha\beta} \,-\, \Lambda^{(3)}(g_1, g_2)^{\alpha} \,+\, \Lambda^{(3)}(g_1, g_2)^{\beta} \\
& \qquad \qquad \,-\, d\log \lambda^{g_1, g_2}_{\alpha\beta} \\
\Lambda^{(3)}(g_2, g_3)^{\alpha} \,+\, \Lambda^{(3)}(g_1, g_2 g_3)^{\alpha} &= g_3^* \Lambda^{(3)}(g_1, g_2)^{\alpha} \,+\, \Lambda^{(3)}(g_1 g_2, g_3)^{\alpha} \,+\, d\log \gamma^{g_1, g_2, g_3}_{\alpha}
\end{aligned}
$$

$$
\begin{aligned}
\nu^{g_1 g_2}_{\alpha\beta\gamma} &= \left(\nu^{g_2}_{\alpha\beta\gamma} \right) \left(g_2^* \nu^{g_1}_{\alpha\beta\gamma} \right) \left(\lambda^{g_1, g_2}_{\alpha\beta} \right) \left(\lambda^{g_1, g_2}_{\beta\gamma} \right) \left(\lambda^{g_1, g_2}_{\gamma\alpha} \right) \\
\left(\lambda^{g_1 g_2, g_3}_{\alpha\beta} \right) \left(g_3^* \lambda^{g_1, g_2}_{\alpha\beta} \right) &= \left(\lambda^{g_1, g_2 g_3}_{\alpha\beta} \right) \left(\lambda^{g_2, g_3}_{\alpha\beta} \right) \left(\gamma^{g_1, g_2, g_3}_{\alpha} \right) \left(\gamma^{g_1, g_2, g_3}_{\beta} \right)^{-1} \\
\left(\gamma^{g_1, g_2, g_3 g_4}_{\alpha} \right) \left(\gamma^{g_1 g_2, g_3, g_4}_{\alpha} \right) &= \left(\gamma^{g_1, g_2 g_3, g_4}_{\alpha} \right) \left(\gamma^{g_2, g_3, g_4}_{\alpha} \right) \left(g_4^* \gamma^{g_1, g_2, g_3}_{\alpha} \right)
\end{aligned}
$$

where $\nu^g_{\alpha\beta\gamma}$, $\lambda^{g_1,g_2}_{\alpha\beta}$, $\gamma^{g_1,g_2,g_3}_{\alpha}$, $\Lambda^{(1)}(g)^{\alpha\beta}$, $\Lambda^{(2)}(g)^{\alpha}$, and $\Lambda^{(3)}(g_1,g_2)^{\alpha}$ are structures introduced to define the orbifold group action.

2.2 Differences between orbifold group actions

In describing orbifold $U(1)$ Wilson lines, the group $H^1(\Gamma, U(1))$ arises as differences between orbifold group actions; similarly, in describing discrete torsion, $H^2(\Gamma, U(1))$ arises in describing the differences between orbifold group actions. Here, we shall also study differences between orbifold group actions, and, at the end of the day, we shall discover that some of the differences between orbifold group actions are classified by $H^3(\Gamma, U(1))$.

Let one lift of the orbifold group Γ be denoted using the same notation as in the last section, and let a second lift be denoted by overlining.

To begin, define

$$
\Upsilon^g_{\alpha\beta\gamma} \;=\; \frac{\nu^g_{\alpha\beta\gamma}}{\overline{\nu}^g_{\alpha\beta\gamma}}
$$

From equation (1), as applied to each of the two lifts, it is straightforward to derive that the Υ are Čech cocycles:

$$
\delta \Upsilon^g \;=\; 1 \tag{13}
$$

In fact, such Čech cocycles define a 1-gerbe. This should not be surprising – after all, a gauge transformation of a 1-gerbe is a 0-gerbe, i.e., a principal $U(1)$ bundle. Here, we see that the

difference between two lifts of orbifold group actions to 2-gerbes is a gauge transformation of the 2-gerbe, namely, a set of 1-gerbes.

Define

$$\begin{aligned}\mathcal{B}(g)^{\alpha} &= \Lambda^{(2)}(g)^{\alpha} - \overline{\Lambda}^{(2)}(g)^{\alpha} \\ \mathcal{A}(g)^{\alpha\beta} &= \overline{\Lambda}^{(1)}(g)^{\alpha\beta} - \Lambda^{(1)}(g)^{\alpha\beta}\end{aligned}$$

By comparing $g^* A^{\alpha\beta\gamma}$ expressed in terms of the two different lifts, we find that

$$\mathcal{A}(g)^{\alpha\beta} + \mathcal{A}(g)^{\beta\gamma} + \mathcal{A}(g)^{\gamma\alpha} = d\log \Upsilon^{g}_{\alpha\beta\gamma} \tag{14}$$

and by comparing $g^* B^{\alpha\beta}$ expressed in terms of the two different lifts, we find that

$$\mathcal{B}(g)^{\alpha} - \mathcal{B}(g)^{\beta} = d\mathcal{A}(g)^{\alpha\beta} \tag{15}$$

From equations (15), (14), and (13) we see that the data $(\mathcal{B}(g)^{\alpha}, \mathcal{A}(g)^{\alpha\beta}, \Upsilon^{g}_{\alpha\beta\gamma})$ defines a 1-gerbe with connection, in the language of [12, 13]. Again, as mentioned a few paragraphs ago, a 1-gerbe with connection defines a gauge transformation, so we have just learned that (part of) the difference between any two lifts of the orbifold group action is a set of gauge transformations, one for each element of the orbifold group. This is precisely analogous to orbifold $U(1)$ Wilson lines, where we observed that any two lifts of the action of the orbifold group differ by a set of gauge transformations.

Furthermore, from comparing $g^* C^{\alpha}$ expressed in terms of the two different lifts, we find that

$$d\mathcal{B}(g)^{\alpha} = 0 \tag{16}$$

In other words, to preserve the connection on the 2-gerbe, the possible gauge transformations are restricted to 1-gerbes with flat connection, just as for orbifold $U(1)$ Wilson lines, to preserve the connection, gauge transformations were restricted to be constant.

Of course, there is additional structure present here. Define

$$\Omega^{g_1,g_2}_{\alpha\beta} = \frac{\lambda^{g_1,g_2}_{\alpha\beta}}{\overline{\lambda}^{g_1,g_2}_{\alpha\beta}}$$

From dividing the two expressions

$$\begin{aligned}\nu^{g_1 g_2}_{\alpha\beta\gamma} &= \left(\nu^{g_2}_{\alpha\beta\gamma}\right)\left(g_2^* \nu^{g_1}_{\alpha\beta\gamma}\right)\left(\lambda^{g_1,g_2}_{\alpha\beta}\right)\left(\lambda^{g_1,g_2}_{\beta\gamma}\right)\left(\lambda^{g_1,g_2}_{\gamma\alpha}\right) \\ \overline{\nu}^{g_1 g_2}_{\alpha\beta\gamma} &= \left(\overline{\nu}^{g_2}_{\alpha\beta\gamma}\right)\left(g_2^* \overline{\nu}^{g_1}_{\alpha\beta\gamma}\right)\left(\overline{\lambda}^{g_1,g_2}_{\alpha\beta}\right)\left(\overline{\lambda}^{g_1,g_2}_{\beta\gamma}\right)\left(\overline{\lambda}^{g_1,g_2}_{\gamma\alpha}\right)\end{aligned}$$

we see that

$$\Upsilon^{g_1 g_2}_{\alpha\beta\gamma} = \left(\Upsilon^{g_2}_{\alpha\beta\gamma}\right)\left(g_2^* \Upsilon^{g_1}_{\alpha\beta\gamma}\right)\left(\Omega^{g_1,g_2}_{\alpha\beta}\right)\left(\Omega^{g_1,g_2}_{\beta\gamma}\right)\left(\Omega^{g_1,g_2}_{\gamma\alpha}\right) \tag{17}$$

which is the statement in local trivializations [6] that Ω^{g_1,g_2} defines a map between 1-gerbes. (We should be slightly careful – although the Ω^{g_1,g_2} define a map between 1-gerbes, they are not sufficient to completely define a map between 1-gerbes with connection. In a few paragraphs we will find the additional structure needed to complete the maps Ω^{g_1,g_2} to define a morphism of 1-gerbes with connection.) Denoting the 1-gerbes described by the cocycles $\Upsilon^g_{\alpha\beta\gamma}$ by Υ^g, we can describe the maps Ω^{g_1,g_2} defined by the Čech cochains $\Omega^{g_1,g_2}_{\alpha\beta}$ as maps

$$\Omega^{g_1,g_2} : \; \Upsilon^{g_2} \otimes g_2^* \Upsilon^{g_1} \; \longrightarrow \; \Upsilon^{g_1 g_2}$$

We should take a moment to explain the meaning of $\otimes$ for 1-gerbes. A technical definition is given in [14, section 4.1]. However, for the purposes of most readers, it should suffice to think of $\otimes$ as an operation which associates a 1-gerbe to Čech cocycles given by the product of Čech cocycles defining two other 1-gerbes.

The constraint (3) becomes the constraint that there is a well-defined relationship between $\Upsilon^{g_1 g_2 g_3}$ and $\Upsilon^{g_3} \otimes g_3^* \left(\Upsilon^{g_2} \otimes g_2^* \Upsilon^{g_1} \right)$. As we shall explain more concretely in a moment, constraint (3) implies that the following diagram[2] commutes, up to isomorphism of maps:

$$\begin{array}{ccc} \Upsilon^{g_3} \otimes g_3^* \left(\Upsilon^{g_2} \otimes g_2^* \Upsilon^{g_1} \right) & \stackrel{g_3^* \Omega^{g_1,g_2}}{\longrightarrow} & \Upsilon^{g_3} \otimes g_3^* \Upsilon^{g_1 g_2} \\ {\scriptstyle \Omega^{g_2,g_3}} \downarrow & & \downarrow {\scriptstyle \Omega^{g_1 g_2, g_3}} \\ \Upsilon^{g_2 g_3} \otimes (g_2 g_3)^* \Upsilon^{g_1} & \stackrel{\Omega^{g_1, g_2 g_3}}{\longrightarrow} & \Upsilon^{g_1 g_2 g_3} \end{array} \tag{18}$$

Again, the correct statement is that the diagram above commutes up to isomorphism of maps:

$$\Omega^{g_1 g_2, g_3} \circ g_3^* \Omega^{g_1,g_2} \; \cong \; \Omega^{g_1, g_2 g_3} \circ \Omega^{g_2,g_3}$$

To understand why these maps are merely isomorphic – or what a map between maps is, in this context – recall from [6] that 1-gerbes can be understood as sheaves of categories, roughly speaking, and maps between them are sheaves of functors. A "map between maps," in this context, is a sheaf of natural transformations.

More concretely, these "maps between maps" are defined by the Čech cochains γ^{g_1,g_2,g_3}. Define

$$\omega^{g_1,g_2,g_3}_\alpha \; = \; \frac{ \gamma^{g_1,g_2,g_3}_\alpha }{ \overline{\gamma}^{g_1,g_2,g_3}_\alpha }$$

From the analogues of equation (3) for the two orbifold group actions, we see that each Čech cochain $\omega^{g_1,g_2,g_3}_\alpha$ obeys

$$\left(\Omega^{g_1 g_2, g_3}_{\alpha\beta} \right) \left(g_3^* \Omega^{g_1,g_2}_{\alpha\beta} \right) \; = \; \left(\Omega^{g_1, g_2 g_3}_{\alpha\beta} \right) \left(\Omega^{g_2,g_3}_{\alpha\beta} \right) \left(\omega^{g_1,g_2,g_3}_\alpha \right) \left(\omega^{g_1,g_2,g_3}_\beta \right)^{-1} \tag{19}$$

[2] Experts will recognize that we are being slightly sloppy, in that, for example, we identify $(g_2 g_3)^* \Upsilon^{g_1}$ with $g_3^* g_2^* \Upsilon^{g_1}$. Strictly speaking, for 1-gerbes the story is slightly more complicated, in that they are merely canonically isomorphic. However, at the level of Čech cocycles, such technical distinctions are not present, so we omit them from the discussion.

This means that the cochains $\omega_\alpha^{g_1,g_2,g_3}$ define a map ω^{g_1,g_2,g_3} between maps:

$$\omega^{g_1,g_2,g_3} : \Omega^{g_1,g_2g_3} \circ \Omega^{g_2,g_3} \longrightarrow \Omega^{g_1g_2,g_3} \circ g_3^* \Omega^{g_1,g_2}$$

From equation (4), we find that the ω are constrained as

$$\left(\omega_\alpha^{g_1,g_2,g_3g_4} \right) \left(\omega_\alpha^{g_1g_2,g_3,g_4} \right) \; = \; \left(\omega_\alpha^{g_1,g_2g_3,g_4} \right) \left(\omega_\alpha^{g_2,g_3,g_4} \right) \left(g_4^* \omega_\alpha^{g_1,g_2,g_3} \right) \tag{20}$$

As a consistency check, the attentive reader will note that both $\omega^{g_1g_2,g_3,g_4} \circ \omega^{g_1,g_2,g_3g_4}$ and $g_4^* \omega^{g_1,g_2,g_3} \circ \omega^{g_1,g_2g_3,g_4} \circ \omega^{g_2,g_3,g_4}$ map

$$\Omega^{g_1,g_2g_3g_4} \circ \Omega^{g_2,g_3g_4} \circ \Omega^{g_3,g_4} \longrightarrow \Omega^{g_1g_2g_3,g_4} \circ g_4^* \Omega^{g_1g_2,g_3} \circ (g_3g_4)^* \Omega^{g_1,g_2}$$

From equation (20), we see that the two possible maps are the same.

Finally, define

$$\theta(g_1,g_2)^\alpha \; = \; \overline{\Lambda}^{(3)}(g_1,g_2)^\alpha \; - \; \Lambda^{(3)}(g_1,g_2)^\alpha \tag{21}$$

We shall see concretely in a moment that $\theta(g_1,g_2)$ combines with Ω^{g_1,g_2} to define a map of 1-gerbes with connection (whereas Ω^{g_1,g_2} by itself merely defined a map of 1-gerbes, and was not sufficient to describe the action on the connection).

From equation (10) for each orbifold group action, we immediately see that

$$\mathcal{B}(g_1g_2)^\alpha \; = \; \mathcal{B}(g_2)^\alpha \; + \; g_2^* \mathcal{B}(g_1)^\alpha \; - \; d\theta(g_1,g_2)^\alpha \tag{22}$$

From equation (11) for each orbifold group action, we find that

$$\mathcal{A}(g_1g_2)^{\alpha\beta} \; = \; \mathcal{A}(g_2)^{\alpha\beta} \; + \; g_2^* \mathcal{A}(g_1)^{\alpha\beta} \; - \; \theta(g_1,g_2)^\alpha \; + \; \theta(g_1,g_2)^\beta \; + \; d \log \Omega_{\alpha\beta}^{g_1,g_2} \tag{23}$$

The two equations above mean that the pair $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$ define a map of 1-gerbes with connection:

$$\left(\mathcal{B}(g_2)^\alpha, \mathcal{A}(g_2)^{\alpha\beta}, \Upsilon_{\alpha\beta\gamma}^{g_2} \right) \otimes g_2^* \left(\mathcal{B}(g_1)^\alpha, \mathcal{A}(g_1)^{\alpha\beta}, \Upsilon_{\alpha\beta\gamma}^{g_1} \right) \longrightarrow \left(\mathcal{B}(g_1g_2)^\alpha, \mathcal{A}(g_1g_2)^{\alpha\beta}, \Upsilon_{\alpha\beta\gamma}^{g_1g_2} \right)$$

From equation (12) for each orbifold group action, we find that

$$\theta(g_2,g_3)^\alpha \; + \; \theta(g_1,g_2g_3)^\alpha \; = \; g_3^* \theta(g_1,g_2)^\alpha \; + \; \theta(g_1g_2,g_3)^\alpha \; - \; d \log \omega_\alpha^{g_1,g_2,g_3} \tag{24}$$

which means that diagram (18) commutes up to isomorphism of maps when interpreted as a diagram of 1-gerbes with connection, not merely gerbes. Put another way, this means that the "map between maps" ω^{g_1,g_2,g_3} is not only a map between the morphisms of 1-gerbes Ω^{g_1,g_2}, but is also a map between the morphisms of 1-gerbes with connection $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$.

To summarize, we have found that the difference between any two orbifold group action on a set of C fields is defined by a set of 1-gerbes with flat connection $(\mathcal{B}(g)^\alpha, \mathcal{A}(g)^{\alpha\beta}, \Upsilon^g_{\alpha\beta\gamma})$, one for each $g \in \Gamma$, plus maps $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$ between 1-gerbes with connection

$$\left(\mathcal{B}(g_2)^\alpha, \mathcal{A}(g_2)^{\alpha\beta}, \Upsilon^{g_2}_{\alpha\beta\gamma}\right) \otimes g_2^* \left(\mathcal{B}(g_1)^\alpha, \mathcal{A}(g_1)^{\alpha\beta}, \Upsilon^{g_1}_{\alpha\beta\gamma}\right) \longrightarrow \left(\mathcal{B}(g_1 g_2)^\alpha, \mathcal{A}(g_1 g_2)^{\alpha\beta}, \Upsilon^{g_1 g_2}_{\alpha\beta\gamma}\right)$$

and "maps between maps" ω^{g_1,g_2,g_3}:

$$\omega^{g_1,g_2,g_3} : \: \Omega^{g_1,g_2 g_3} \circ \Omega^{g_2,g_3} \longrightarrow \Omega^{g_1 g_2,g_3} \circ g_3^* \Omega^{g_1,g_2}$$

The 1-gerbe morphisms $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$ are constrained to make the following diagram commute

$$\begin{array}{ccc} \Upsilon^{g_3} \otimes g_3^* \left(\Upsilon^{g_2} \otimes g_2^* \Upsilon^{g_1} \right) & \stackrel{g_3^* \Omega^{g_1,g_2}}{\longrightarrow} & \Upsilon^{g_3} \otimes g_3^* \Upsilon^{g_1 g_2} \\ \scriptstyle{\Omega^{g_2,g_3}} \downarrow & & \downarrow \scriptstyle{\Omega^{g_1 g_2,g_3}} \\ \Upsilon^{g_2 g_3} \otimes (g_2 g_3)^* \Upsilon^{g_1} & \stackrel{\Omega^{g_1,g_2 g_3}}{\longrightarrow} & \Upsilon^{g_1 g_2 g_3} \end{array}$$

up to isomorphisms of maps defined by ω^{g_1,g_2,g_3}, and the maps of maps ω^{g_1,g_2,g_3} are constrained to obey

$$\omega^{g_1 g_2,g_3,g_4} \circ \omega^{g_1,g_2,g_3 g_4} \: = \: g_4^* \omega^{g_1,g_2,g_3} \circ \omega^{g_1,g_2 g_3,g_4} \circ \omega^{g_2,g_3,g_4}$$

2.3 Residual gauge invariances

Just as in [5, 6, 7], there are residual gauge invariances. Put another way, because under a gauge transformation, $C \mapsto C + dB$, if two 1-gerbe connections B differ by a 1-gerbe gauge transformation, then they define the same action on C. Put another way still, only equivalence classes of 1-gerbes define distinct gauge transformations of a C field. In addition, the maps between the 1-gerbes also have residual gauge transformations. For example, in the next section we shall see that if the 1-gerbes Υ^g are topologically trivial, then the maps Ω^{g_1,g_2} become bundles, defining gauge transformations of the Υ^g. However, just as in [5, 6, 7], only equivalence classes of bundles with connection define distinct gauge transformations of a 1-gerbe, so we have a second level of residual gauge invariances.

In the next two subsections we shall work out these two classes of residual gauge invariances in more detail.

2.3.1 First level of residual gauge invariances

The first level of residual gauge invariances that we consider consists of replacing one set of 1-gerbes with connection (partially defining the difference between two orbifold group actions) with an isomorphic set of 1-gerbes with connection. In other words, since only equivalence classes of 1-gerbes with connection act on a 2-gerbe with connection, how does

the data defining the difference between two orbifold group actions change when the 1-gerbes are replaced with isomorphic 1-gerbes?

Suppose $(\kappa^g, \chi(g)^\alpha)$ defines a map between two isomorphic 1-gerbes with connection:

$$(\kappa^g, \chi(g)^\alpha) : \left(\Upsilon^g, \mathcal{B}(g)^\alpha, \mathcal{A}(g)^{\alpha\beta}\right) \longrightarrow \left(\overline{\Upsilon}^g, \overline{\mathcal{B}}(g)^\alpha, \overline{\mathcal{A}}(g)^{\alpha\beta}\right)$$

In other words, at the level of Čech cochain data, suppose that the data defining the two sets of 1-gerbes with connection is related as follows:

$$\begin{aligned}
\overline{\Upsilon}^g_{\alpha\beta\gamma} &= \left(\Upsilon^g_{\alpha\beta\gamma}\right)\left(\kappa^g_{\alpha\beta}\right)\left(\kappa^g_{\beta\gamma}\right)\left(\kappa^g_{\gamma\alpha}\right) \\
\overline{\mathcal{B}}(g)^\alpha &= \mathcal{B}(g)^\alpha - d\chi(g)^\alpha \\
\overline{\mathcal{A}}(g)^{\alpha\beta} &= \mathcal{A}(g)^{\alpha\beta} - \chi(g)^\alpha + \chi(g)^\beta + d\log\kappa^g_{\alpha\beta}
\end{aligned}$$

Recall that in order to define an orbifold group action, in addition to specifying 1-gerbes with connection, we also must specify maps $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$ between the 1-gerbes with connection, as well as maps ω^{g_1,g_2,g_3} between the maps between the maps between the 1-gerbes with connection. If we replace one set of 1-gerbes with connection with an isomorphic set of 1-gerbes with connection as above, then it is straightforward to check that the remaining data defining the difference between orbifold group actions changes as follows:

$$\begin{aligned}
\overline{\Omega}^{g_1,g_2}_{\alpha\beta} &= \left(\Omega^{g_1,g_2}_{\alpha\beta}\right)\left(\kappa^{g_1 g_2}_{\alpha\beta}\right)\left(\kappa^{g_2}_{\alpha\beta}\right)^{-1}\left(g_2^*\kappa^{g_1}_{\alpha\beta}\right)^{-1} \\
\overline{\theta}(g_1,g_2)^\alpha &= \theta(g_1,g_2)^\alpha + \chi(g_1 g_2)^\alpha - \chi(g_2)^\alpha - g_2^*\chi(g_1)^\alpha \\
\overline{\omega}^{g_1,g_2,g_3}_\alpha &= \omega^{g_1,g_2,g_3}_\alpha
\end{aligned}$$

In short, the maps $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$ are changed when the 1-gerbes are replaced with isomorphic 1-gerbes with connection, but the maps between maps (ω) remain invariant.

2.3.2 Second level of residual gauge invariances

Even if we hold fixed the 1-gerbes with connection $(\Upsilon^g, \mathcal{B}(g), \mathcal{A}(g))$, we still have a remaining residual gauge invariance, produced by replacing the maps $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$ with isomorphic maps.

Let κ^{g_1,g_2} define a map between two such maps, as

$$\kappa^{g_1,g_2} : \left(\Omega^{g_1,g_2}, \theta(g_1,g_2)\right) \longrightarrow \left(\overline{\Omega}^{g_1,g_2}, \overline{\theta}(g_1,g_2)\right)$$

At the level of Čech cochains we can describe this map as

$$\begin{aligned}
\overline{\Omega}^{g_1,g_2}_{\alpha\beta} &= \left(\Omega^{g_1,g_2}_{\alpha\beta}\right)\left(\kappa^{g_1,g_2}_\alpha\right)\left(\kappa^{g_1,g_2}_\beta\right)^{-1} \\
\overline{\theta}(g_1,g_2)^\alpha &= \theta(g_1,g_2)^\alpha + d\log\kappa^{g_1,g_2}_\alpha
\end{aligned}$$

It is straightforward to check that if we change $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$ as above, then we must also alter the maps between maps as

$$\overline{\omega}_\alpha^{g_1,g_2,g_3} = \left(\omega_\alpha^{g_1,g_2,g_3}\right) \left(\kappa_\alpha^{g_1g_2,g_3}\right) \left(g_3^* \kappa_\alpha^{g_1,g_2}\right) \left(\kappa_\alpha^{g_1,g_2g_3}\right)^{-1} \left(\kappa_\alpha^{g_2,g_3}\right)^{-1}$$

Formally, if both the ω and κ were constant maps, note that the equation above would amount to the statement that $\overline{\omega}$ and ω differ by a group 3-coboundary defined by κ.

2.4 $H^3(\Gamma, U(1))$

At least part (and sometimes all) of the possible differences between orbifold group actions are described by elements of the group cohomology group[3] $H^3(\Gamma, U(1))$, as we shall now describe.

Recall from [6, 7] that in order to find $H^2(\Gamma, U(1))$ in data consisting of a set of bundles T^g with connection, together with isomorphisms $\omega^{g,h}$, one took the bundles to be topologically trivial, and the connections on those bundles to be gauge-trivial. The resulting degrees of freedom resided essentially entirely in the isomorphisms $\omega^{g,h}$, and were counted by $H^2(\Gamma, U(1))$. Also, because the bundles were trivial, and the gauge-connections were trivial, Wilson surfaces $\exp\left(\int B\right)$ around Riemann surfaces created by the orbifold group action had a constant phase, independent of the details of the Riemann surface, depending only on the orbifold group elements used to glue together the sides.

We find $H^3(\Gamma, U(1))$ in the present case in a similarly restrictive context. Take the 1-gerbes Υ^g to be canonically trivial (meaning, all transition functions identically 1), and assume all the $\mathcal{A}(g)^{\alpha\beta} = 0$, so the B field is a globally-defined 2-form. (Equivalently, take the 1-gerbes Υ^g to be trivial with gauge-trivial connection; then we can map to the situation just described to get an equivalent action on the C field.)

In this case, the 1-gerbe maps Ω^{g_1,g_2} become principal $U(1)$ bundles, from equation (17). (This is just another way of saying that a 1-gerbe map from a gerbe into itself is a gauge transformation of the 1-gerbe, and such gauge transformations are defined by bundles.) Also, from equation (23) we see that $\theta(g_1,g_2)^\alpha$ define a connection on the principal $U(1)$ bundle Ω^{g_1,g_2}. Also, from equation (24) we see that the bundle morphism ω^{g_1,g_2,g_3} preserves the connection on the bundles.

Next, in order to find $H^3(\Gamma, U(1))$, we need to make another restriction. Take the bundles Ω^{g_1,g_2} to all be topologically trivial, and the connections $\theta(g_1,g_2)$ to all be gauge-trivial.

Again, only equivalence classes of bundles with connection are meaningful, so this data can be equivalently mapped to the bundles Ω^{g_1,g_2} being canonically trivial and the connec-

[3] With trivial action on the coefficients.

tions $\theta(g_1, g_2)$ identically zero. We see from equation (24) that the maps ω^{g_1,g_2,g_3} must be constant.

Assuming the covering space is connected, the maps ω^{g_1,g_2,g_3} define a map $\Gamma \times \Gamma \times \Gamma \to U(1)$. Also, from equation (20) we see that these maps satisfy the group 3-cocycle condition. Finally, recall that we still have a residual gauge invariance: we can gauge-transform the bundles Ω^{g_1,g_2} by a constant gauge transformation, as described in section 2.3.2. It is easy to see that such gauge transformations change the maps ω^{g_1,g_2,g_3} by a group 3-coboundary.

Thus, we find that (many) differences between orbifold group actions are classified by elements of the group cohomology group $H^3(\Gamma, U(1))$, where the group 3-cocycles ultimately come from the maps ω^{g_1,g_2,g_3}.

As a check, it is straightforward to show that the group $H^3(\Gamma, U(1))$ also enters cohomology calculations in simple cases. As a simple example, consider possible holonomies of a 3-form potential on an n-torus T^n. Construct T^n as a quotient: $T^n = \mathbf{R}^n/\mathbf{Z}^n$. Now, [15, section III.1]

$$H_i(\mathbf{Z}, \mathbf{Z}) \;=\; \begin{cases} \mathbf{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases}$$

so from the Künneth formula [15, section V.5] we find that

$$H_3(\mathbf{Z}^n, \mathbf{Z}) \;=\; \mathbf{Z}^k$$

where

$$k \;=\; \left(\begin{array}{c} n \\ 3 \end{array} \right)$$

and from the universal coefficient theorem [15, section III.1 ex. 3]

$$\begin{aligned} H^3(\mathbf{Z}^n, U(1)) &= \mathrm{Hom}\left(H_3(\mathbf{Z}^n, \mathbf{Z}), U(1)\right) \\ &= U(1)^k \end{aligned}$$

so we find that if we orbifold the trivial C field (i.e., $C \equiv 0$) on $\mathbf{R}^n$ by the freely-acting $\mathbf{Z}^n$, the number of possible ways to combine the action of the orbifold group with a gauge transformation of the C fields is counted by

$$[U(1)]^k$$

which happily coincides with the possible holonomies of flat (and topologically trivial) C fields on T^n. Thus, we should not be surprised to find the group $H^3(\Gamma, U(1))$ appearing in our calculation of analogues of discrete torsion for the C field.

2.5 Detailed classification of orbifold group actions

In the last subsection we described how the differences between many orbifold group actions were described by elements of $H^3(\Gamma, U(1))$. Under what circumstances do those represent all of the differences, and what do additional differences look like?

Let X denote the covering space. Suppose $H^3(X, \mathbf{Z})$ has no torsion, $H^2(X, \mathbf{Z}) = 0$, and $\pi_1(X) = 0$. Then all orbifold group actions on C fields differ by an element of $H^3(\Gamma, U(1))$. This is straightforward to check. Since $H^3(X, \mathbf{Z})$ contains no torsion, all the flat 1-gerbes Υ^g must be topologically trivial, and so can be mapped to the canonical trivial 1-gerbe. Since $H^2(X, \mathbf{Z}) = 0$, any connection on the 1-gerbes Υ^g must be gauge-trivial, and so we can map them all to the zero connection, without loss of generality. The 1-gerbe maps $(\Omega^{g_1,g_2}, \theta(g_1, g_2))$ are now bundles with connection. However, because $H^2(X, \mathbf{Z}) = 0$ and $\pi_1(X) = 0$, the bundles Ω^{g_1,g_2} must be topologically trivial, and the connection $\theta(g_1, g_2)$ must be gauge-trivial.

More generally, however, one can expect to find additional possible orbifold group actions, just as for B fields in [7], *i.e.* analogues for C fields of shift orbifolds [16]. This is certainly consistent with naive examinations of homology. From the Cartan-Leray spectral sequence [15, section VII.7]:

$$E^2_{p,q} \;=\; H_p\left(\Gamma, H_q(X, \mathbf{Z})\right) \;\Longrightarrow\; H_{p+q}(X/\Gamma, \mathbf{Z})$$

Ignoring differentials for a moment, we can see immediately that the homology $H_3(X/\Gamma, \mathbf{Z})$ involves more than just $H_3(X, \mathbf{Z})$ and $H_3(\Gamma, \mathbf{Z})$ (which dualizes to $H^3(\Gamma, U(1))$; there can also be nontrivial contributions involving $H_1(X, \mathbf{Z})$ and $H_2(X, \mathbf{Z})$.

In the remainder of this article, we shall concentrate on the differences between orbifold group actions classified by $H^3(\Gamma, U(1))$, and will largely ignore the additional possible orbifold group actions. We will return to these additional degrees of freedom in [16], where we shall argue that they encode the M-theory dual of IIA discrete torsion.

2.6 Commentary

In this section, we have studied orbifold group actions on C fields, and shown, for example, that the group $H^3(\Gamma, U(1))$ arises in describing the differences between (some) orbifold group actions.

We should point out that our analysis does not depend upon whether or not the orbifold group Γ acts freely, just as our analysis of B fields in [5, 6, 7] did not depend upon whether Γ acts freely. (Now, understanding M theory on singular spaces can be somewhat subtle, so there may be physics subtleties; however, our mathematical analysis is independent of whether or not Γ acts freely.)

We should also point out that our analysis does not depend upon whether or not Γ is abelian, just as our analysis of B fields in [5, 6, 7] did not depend upon whether Γ is abelian.

We should also mention that our analysis does not depend upon the curvature of the C field being zero in integral cohomology $H^4(\mathbf{Z})$, just as for our analysis of B fields in [5, 6, 7]. However, if our 2-gerbe is topologically nontrivial, then we need to check that an action of the orbifold group exists – just as for vector fields and B fields, topologically nontrivial objects do not always admit orbifold group actions. The analysis we have provided continues to hold on the assumption that orbifold group actions exist.

3 Membrane twisted sector phases

Discrete torsion (for B fields) was originally discovered in [1] as a phase ambiguity in twisted sector contributions to string loop partition functions. As noted in [5, 7], this phase ambiguity arises because of the presence of a term $\int B$ in the worldsheet sigma model.

Now, low-energy effective actions for membrane worldvolumes contain a term $\int C$, so one would expect that "membrane twisted sectors" would be weighted by factors derived from $\exp\left(\int B\right)$.

In this section we shall derive these membrane worldvolume factors, the analogue of the twisted sector phases of [1] for membranes. We shall also check that these factors are invariant under the natural $SL(3, \mathbf{Z})$ action on T^3.

At this point we should emphasize that we do not wish to imply that we believe that M theory is a theory of membranes; rather, we are merely computing how the low-energy effective action on a membrane worldvolume sees our degrees of freedom. We should also reiterate that our analysis is too naive physically in that we have ignored gravitational corrections and underlying E_8 structures in the M-theory C form [9, 10], and instead blindly treated the C form as a connection on a 2-gerbe.

3.1 Twisted sector phases on T^3

For simplicity, we will assume that $C \equiv 0$ (and that the 2-gerbe is topologically trivial), so that any orbifold group action on the C field can be described solely in terms of its difference from the identity.

Furthermore, for simplicity we shall also only consider those orbifold group actions described by elements of $H^3(\Gamma, U(1))$. This means that we shall take the 1-gerbes Υ^g to all be canonically topologically trivial, with flat $\mathcal{B}(g)^\alpha$ and, for additional simplicity, $\mathcal{A}(g)^{\alpha\beta} \equiv 0$.

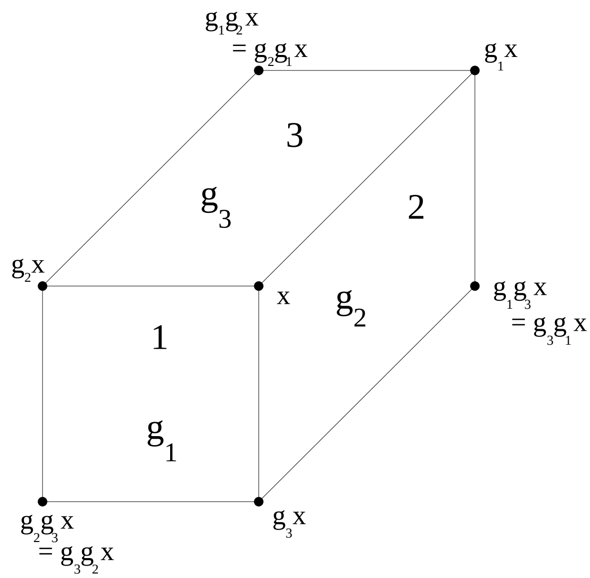

Figure 1: Three-torus seen as open box on covering space.

The gerbe maps $(\Omega^{g_1,g_2}, \theta(g_1,g_2))$ can now be interpreted as principal $U(1)$ bundles with connection. We assume each bundle Ω^{g_1,g_2} is topologically trivial, and that the connection $\theta(g_1,g_2)$ is gauge-trivial.

Consider a T^3 in the quotient space, which looks like a box in the covering space with sides identified by the orbifold group action, as illustrated in figure (1).

Following the procedure described in [7], from glueing the faces of the cube together, one would naively believe that the holonomy

$$\exp\left(\int C \right)$$

over the volume of the cube on the covering space should be corrected by a phase factor

$$\exp\left(\int_1 \mathcal{B}(g_1) \; + \; \int_2 \mathcal{B}(g_2) \; + \; \int_3 \mathcal{B}(g_3) \right) \tag{25}$$

However, just as in [7], this can not be the complete answer, if for no other reason than it is not invariant under gauge transformations of the 1-gerbes. We can go a long way towards fixing this difficulty by closely examining the edges at the borders of each face.

Consider the edges shown in figure (2), labelled a, b, $g_2^* a$, and $g_1^* b$, between faces 1 and 2. These edges are all mapped to the same curve in the quotient space by the orbifold group action. From the expression

$$\mathcal{B}(g_1 g_2)^\alpha \;=\; \mathcal{B}(g_2)^\alpha \;+\; g_2^* \mathcal{B}(g_1)^\alpha \;-\; d\theta(g_1,g_2)^\alpha$$

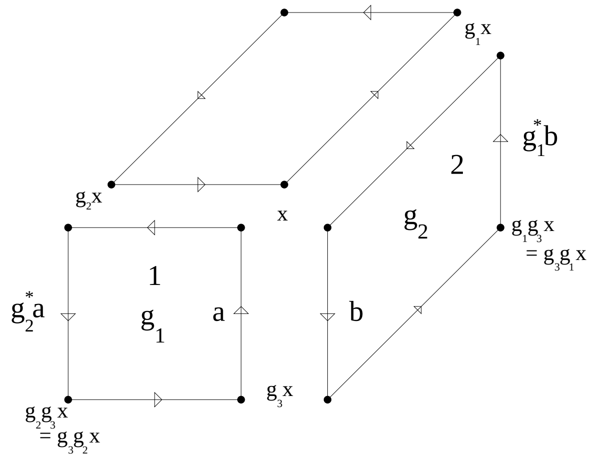

Figure 2: Four of the twelve edges, all descending to same line.

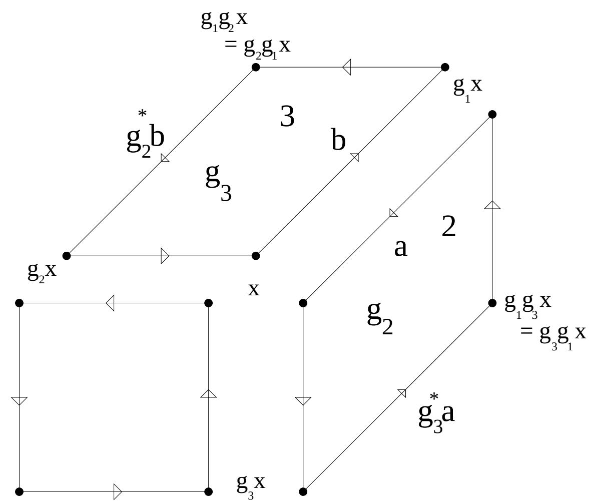

Figure 3: Four of the twelve edges, all descending to same line.

we can derive that

$$[\,\mathcal{B}(g_2)\ +\ g_2^*\mathcal{B}(g_1)\,]\ -\ [\,\mathcal{B}(g_1)\ +\ g_1^*\mathcal{B}(g_2)\,]\ =\ d\,(\,\theta(g_1,g_2)\ -\ \theta(g_2,g_1)\,)$$

so we can partially fix the naive expression (25) by adding the factor

$$\exp\left(-\int_x^{g_3x}[\,\theta(g_1,g_2)\ -\ \theta(g_2,g_1)\,]\right) \tag{26}$$

to take into account these four edges.

Next, consider the edges shown in figure (3), labelled as before, between faces 2 and 3. Proceeding as above, we find that these edges contribute a factor to expression (25) given

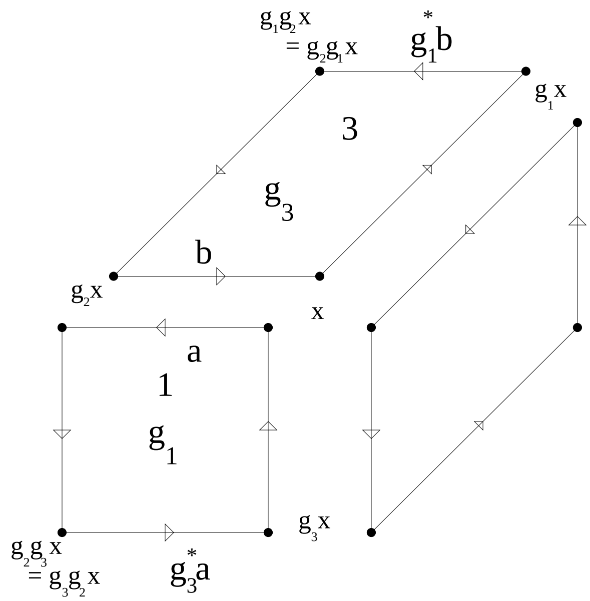

Figure 4: Four of the twelve edges, all descending to same line.

by

$$\exp\left(-\int_{x}^{g_1 x}\left[\,\theta(g_2,g_3)\;-\;\theta(g_3,g_2)\,\right]\right) \tag{27}$$

Finally, consider the remaining four edges shown in figure (4), labelled as before, between faces 1 and 3. Proceeding as above, we find that these edges contribute a factor to expression (25) given by

$$\exp\left(-\int_{g_2 x}^{x}\left[\,\theta(g_1,g_3)\;-\;\theta(g_3,g_1)\,\right]\right) \tag{28}$$

To summarize our results so far, we have found a phase factor (induced by gauge transformations at the boundaries) given by

$$\exp\left(-\int_{x}^{g_3 x}\left[\,\theta(g_1,g_2)\;-\;\theta(g_2,g_1)\,\right]\;-\;\int_{x}^{g_1 x}\;\left[\,\theta(g_2,g_3)\;-\;\theta(g_3,g_2)\,\right]\right)$$
$$\cdot\exp\left(-\int_{g_2 x}^{x}\;\left[\,\theta(g_1,g_3)\;-\;\theta(g_3,g_1)\,\right]\right)\;\cdot\;\exp\left(\int_1 \mathcal{B}(g_1)\;+\;\int_2 \mathcal{B}(g_2)\;+\;\int_3 \mathcal{B}(g_3)\right)$$

However, the phase factor above can still not be completely correct, because it is not invariant under gauge transformations of the bundles Ω^{g_1,g_2}. We need to add an additional factor to fix the contributions from corners of the box.

In order to determine which additional factor to add, note that the connections θ at the ends of the line integrals (i.e., the corners of the box) are given by

$$g_3^*\theta(g_1,g_2)\;-\;g_3^*\theta(g_2,g_1)\;-\;\theta(g_1,g_2)\;+\;\theta(g_2,g_1)$$

$$+\, g_1^*\theta(g_2,g_3) \;-\; g_1^*\theta(g_3,g_2) \;-\; \theta(g_2,g_3) \;+\; \theta(g_3,g_2)$$
$$-\, g_2^*\theta(g_1,g_3) \;+\; g_2^*\theta(g_3,g_1) \;+\; \theta(g_1,g_3) \;-\; \theta(g_3,g_1)$$

Using the identity

$$\theta(g_2,g_3) \;+\; \theta(g_1,g_2g_3) \;=\; g_3^*\theta(g_1,g_2) \;+\; \theta(g_1g_2,g_3) \;-\; d\log\omega^{g_1,g_2,g_3}$$

it is straightforward to check that the twelve-term corner contribution sum above is equal to

$$d\left[\log\omega^{g_1,g_2,g_3} \;-\; \log\omega^{g_2,g_1,g_3} \;-\; \log\omega^{g_3,g_2,g_1} \;+\; \log\omega^{g_3,g_1,g_2} \;+\; \log\omega^{g_2,g_3,g_1} \;-\; \log\omega^{g_1,g_3,g_2}\right]$$

Using this fact, we can now write the correct, completely gauge-invariant, phase factor picked up by $\exp\left(\int C\right)$ because of gauge transformations at boundaries:

$$\begin{array}{l}
\left(\omega_x^{g_1,g_2,g_3}\right)\left(\omega_x^{g_2,g_1,g_3}\right)^{-1}\left(\omega_x^{g_3,g_2,g_1}\right)^{-1}\left(\omega_x^{g_3,g_1,g_2}\right)\left(\omega_x^{g_2,g_3,g_1}\right)\left(\omega_x^{g_1,g_3,g_2}\right)^{-1} \\
\quad \cdot\exp\left(-\int_x^{g_3 x}\left[\theta(g_1,g_2) \;-\; \theta(g_2,g_1)\right] \;-\; \int_x^{g_1 x}\left[\theta(g_2,g_3) \;-\; \theta(g_3,g_2)\right]\right) \\
\quad \cdot\exp\left(-\int_{g_2 x}^{x}\left[\theta(g_1,g_3) \;-\; \theta(g_3,g_1)\right]\right) \\
\quad \cdot\exp\left(\int_1 \mathcal{B}(g_1) \;+\; \int_2 \mathcal{B}(g_2) \;+\; \int_3 \mathcal{B}(g_3)\right)
\end{array}$$

Since the expression above is gauge-invariant, we can evaluate it by evaluating it in any convenient gauge. Consider the gauge in which $\mathcal{B}(g)\equiv 0$ for all g, and $\theta(g_1,g_2)\equiv 0$ for all g_1, g_2. Then the ω^{g_1,g_2,g_3} become constant maps into $U(1)$ (assuming the covering space is connected, of course), describing a group 3-cocycle. In this gauge, it is manifest that the expression above is equal to

$$\left(\omega^{g_1,g_2,g_3}\right)\left(\omega^{g_2,g_1,g_3}\right)^{-1}\left(\omega^{g_3,g_2,g_1}\right)^{-1}\left(\omega^{g_3,g_1,g_2}\right)\left(\omega^{g_2,g_3,g_1}\right)\left(\omega^{g_1,g_3,g_2}\right)^{-1} \tag{29}$$

As a check, note that expression (29) is invariant under changing the group 3-cocycles by coboundaries, as indeed it must be in order to give associate a well-defined phase to elements of $H^3(\Gamma, U(1))$.

3.2 "Modular invariance" on T^3

The twisted sector phases originally defining discrete torsion in [1] had the property that they were invariant under the action of the modular group $SL(2,\mathbf{Z})$ of T^2. In this section we shall review that calculation for B field discrete torsion, and then check that the phases we just computed for "membrane twisted sectors" are invariant under the natural $SL(3,\mathbf{Z})$ acting on T^3.

We do not wish to imply that there necessarily exists a precise notion of modular invariance for membranes. Rather, we are merely observing that the $SL(2, \mathbf{Z})$ invariance of the twisted sector phases in [1] has an analogue here, that the "twisted sector phases" seen by the low-energy effective action on a membrane worldvolume are $SL(3, \mathbf{Z})$ invariant. In particular, we have derived these membrane phases from more fundamental considerations, and at the end of the day, we are observing an $SL(3, \mathbf{Z})$ invariance. In other words, unlike the original discrete torsion story of [1], we are not using $SL(3, \mathbf{Z})$ invariance as a starting point, but rather noting it as a consequence.

3.2.1 Review of modular invariance on T^2

Before checking that the membrane twisted sector phases for T^3 are invariant under $SL(3, \mathbf{Z})$, we shall first take a moment to review how this works for standard discrete torsion and twisted sector phases on T^2.

Recall that in a genus one partition function, the phase $\epsilon(g, h)$ associated to a twisted sector determined by the commuting pair (g, h) is given by

$$\epsilon(g, h) \;=\; \left(\omega^{g,h} \right) \left(\omega^{h,g} \right)^{-1}$$

where $\omega^{g,h}$ is the group 2-cocycle describing an element of $H^2(\Gamma, U(1))$.

An element of $SL(2, \mathbf{Z})$ described by the matrix

$$\mathbf{A} \;=\; \left(\begin{array}{cc} a & b \\ c & d \end{array} \right)$$

maps

$$\begin{array}{ccc} g & \mapsto & g^a \, h^b \\ h & \mapsto & g^c \, h^d \end{array}$$

In terms of twisted sector phases, modular invariance is the constraint that

$$\epsilon \left(g^a h^b , \, g^c h^d \right) \;=\; \epsilon(g, h) \tag{30}$$

To prove equation (30), we use the following identities:

1. $\epsilon(g, h) = \epsilon(h, g)^{-1}$ (by inspection)
2. $\epsilon(g_1 g_2, g_3) \;=\; \epsilon(g_1, g_3) \, \epsilon(g_2, g_3)$

(The second identity is proven by manipulating the group cocycle condition and doing some algebra.)

From these identities, it is straightforward to verify

$$\begin{aligned} \epsilon\left(g^a h^b,\, g^c h^d\right) &= \epsilon(g,h)^{\det \mathbf{A}} \\ &= \epsilon(g,h) \end{aligned}$$

and so modular invariance is proven.

3.2.2 Calculation on T^3

Consider a twisted sector determined by the three commuting elements (g_1, g_2, g_3). Under the action of an element $\mathbf{A}$ of $SL(3,\mathbf{Z})$ given by

$$\mathbf{A} = \left(\begin{array}{ccc} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{array} \right)$$

the orbifold group elements (g_1, g_2, g_3) transform as

$$\begin{aligned} g_1 &\mapsto g_1^{a_{11}}\, g_2^{a_{12}}\, g_3^{a_{13}} \\ g_2 &\mapsto g_1^{a_{21}}\, g_2^{a_{22}}\, g_3^{a_{23}} \\ g_3 &\mapsto g_1^{a_{31}}\, g_2^{a_{32}}\, g_3^{a_{33}} \end{aligned}$$

Define $\epsilon(g_1, g_2, g_3)$ to be the twisted sector phase associated to the group 3-cocycle ω^{g_1,g_2,g_3}:

$$\epsilon(g_1,g_2,g_3) = \left(\omega^{g_1,g_2,g_3}\right) \left(\omega^{g_2,g_1,g_3}\right)^{-1} \left(\omega^{g_3,g_2,g_1}\right)^{-1} \left(\omega^{g_3,g_1,g_2}\right) \left(\omega^{g_2,g_3,g_1}\right) \left(\omega^{g_1,g_3,g_2}\right)^{-1}$$

In order for "modular invariance" to hold, i.e., in order for the twisted sector phases to be invariant under $SL(3,\mathbf{Z})$, it had better be the case that

$$\epsilon\left(g_1^{a_{11}}\, g_2^{a_{12}}\, g_3^{a_{13}},\; g_1^{a_{21}}\, g_2^{a_{22}}\, g_3^{a_{23}},\; g_1^{a_{31}}\, g_2^{a_{32}}\, g_3^{a_{33}} \right) = \epsilon(g_1,g_2,g_3) \tag{31}$$

In order to verify equation (31), it is useful to first note the following identities:

1. $\epsilon(g_1, g_2, g_3)$ is antisymmetric under interchange of any two of the group elements – e.g., $\epsilon(g_1,g_2,g_3) = \epsilon(g_2,g_1,g_3)^{-1}$
2. $\epsilon(g_1 g_2, g_3, g_4) = \epsilon(g_1, g_3, g_4)\, \epsilon(g_2, g_3, g_4)$

The first identity is trivial to verify; the second requires a significant amount of algebraic manipulation of the 3-cocycle condition.

Given these identities, it is straightforward to verify that

$$\begin{aligned} \epsilon\left(g_1^{a_{11}} g_2^{a_{12}} g_3^{a_{13}}, \, g_1^{a_{21}} g_2^{a_{22}} g_3^{a_{23}}, \, g_1^{a_{31}} g_2^{a_{32}} g_3^{a_{33}} \right) &= \epsilon(g_1, g_2, g_3)^{\det \mathbf{A}} \\ &= \epsilon(g_1, g_2, g_3) \end{aligned}$$

and so we see that "modular invariance" holds for membranes on T^3.

Again, we do not wish to imply that there necessarily exists a precise physical analogue of modular invariance for membranes. Rather, having calculated the analogues of twisted sector phases from more fundamental considerations, we are merely noting that those phases are $SL(3, \mathbf{Z})$ invariant, a precise analogue of the modular invariance constraint used in [1] to derive the original B field discrete torsion.

4 Notes on local orbifolds

In [7] we spoke very briefly about the possibility of local orbifold degrees of freedom for B field discrete torsion. To review, given a quotient space such as $T^4/\mathbf{Z}_2$, for example, in addition to degrees of freedom obtained by describing the space as a global orbifold, there can sometimes be additional degrees of freedom. For example, from the discussion in [5, 7], there are 2 possible $\mathbf{Z}_2$ orbifold group actions on a trivial line bundle on T^4. However, if we view $T^4/\mathbf{Z}_2$ as a collection of coordinate patches of the form $\mathbf{C}^2/\mathbf{Z}_2$, each patch has a 2-element degree of freedom obtained from possible $\mathbf{Z}_2$ actions on a trivial line bundle on $\mathbf{C}^2$, and when we glue the patches back together, we find more consistent degrees of freedom than just the 2 global ones. The additional degrees of freedom come from describing the orbifold group action on the line bundle locally, rather than globally, and so we refer to them as local orbifold degrees of freedom. A theory with such local orbifold degrees of freedom "turned on" cannot be obtained by a global orbifold, even though the underlying space can be.

We do not claim to have studied possible local orbifold degrees of freedom for analogues of discrete torsion for C fields; however, in this section we merely wish to point out their possible existence.

In particular, readers wishing to describe all possible compactifications of M theory to 7 dimensions, for example, would need to understand not only global degrees of freedom corresponding to analogues of discrete torsion for the M theory three-form potential C, but also possible local orbifold degrees of freedom.

5 Conclusions

In this paper we have described orbifold group actions on C fields, i.e., the analogue of discrete torsion for C fields. As pointed out in [5, 6, 7], discrete torsion (for B fields) is a purely mathematical consequence of understanding orbifold group actions on B fields, and once one understands that orbifold group action on B fields, it is straightforward to work out the correct version for C fields.

As mentioned in the introduction, it would be naive to immediately read physical results into our calculation. We have treated the C field in isolation, but in fact that is certainly naive [8], and one should take into account interaction terms in the theory which we have neglected.

However, it is still very important to emphasize that in principle, an analogue of discrete torsion exists for the M-theory three-form potential C and the other tensor field potentials of string theory. Although the calculation we have presented is naive, it should make the point that some degrees of freedom analogous to discrete torsion exist, and should be studied in greater detail.

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